In order to show the use of the quantity m, suppose the angle of position and distance of a satellite of a planet have been observed with a filar micrometer in the following manner. The wire is laid across the apparent disk of the planet in such a way that this disk is divided into two parts of equal area. The wire therefore always passes through the centre of gravity of the apparent disk, and it is necessary before using the observations for computing orbits to reduce these measures to the true centre of the planet. If we denote by p and s the observed angle of position and the distance of the satellite, and by θ the angle of position of the line of cusps, the errors of p and s arising from the unsymmetrical form of the disk will be,

$$\Delta s = m \,.\, \sin(p-\theta),$$

$$\Delta p = \frac{m}{s} \,.\, \cos(p-\theta),$$

where Δp is expressed in parts of radius.

To complete the solution we have to find the value of θ. This is given from the spherical triangle between the pole of the Equator and the geocentric places of the Sun and the planet. In this triangle the angle at the planet is $90°-\theta$; and if we denote by d the angular distance between the Sun and the planet, and by α, δ, α', δ', the geocentric right ascensions and declinations of the Sun and planet, we shall have,

$$\begin{aligned} \cos d &= \sin\delta \sin\delta' + \cos\delta \cos\delta' \cos(\alpha'-\alpha), \\ \sin d \cos\theta &= \cos\delta \sin(\alpha'-\alpha), \\ \sin d \sin\theta &= \sin\delta \cos\delta' - \cos\delta \sin\delta' \cos(\alpha'-\alpha). \end{aligned}$$

These equations give the angle θ without ambiguity.

In the case of a spheroidal planet, as Jupiter or Saturn, the apparent outline of the planet will be composed of ellipses, but the determination of these ellipses though possible is much more troublesome, and for these planets the correction for defective illumiuation is hardly sensible.

A PROBLEM AND ITS SOLUTION.

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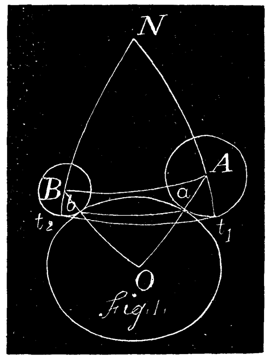
On the surface of a sphere, radius unity, three small circles, radii a, b, c, are described, each touching the other two; on the surface enclosed by them three small circles, radii x_1, y_1, z_1, are described, each touching the other two and two of the first set of circles; on the surface enclosed by the first set of inscribed circles three small circles, radii x_2, y_2, z_2, are described,

each touching the other two and two of the first set of inscribed circles; and so on. Find the radii x_n, y_n, z_n, of the nth set of inscribed circles.

Solution.— We will first prove that, if four small circles each touch a fifth in the same way, and t_1t_2, t_2t_3, t_3t_4, t_4t_1, t_1t_3, t_2t_4 are the arcs of great circles which touch and join the four circles, two and two, we have

$$\sin\tfrac{1}{2}t_1t_2\sin\tfrac{1}{2}t_3t_4+\sin\tfrac{1}{2}t_2t_3\sin\tfrac{1}{2}t_4t_1=\sin\tfrac{1}{2}t_1t_3\sin\tfrac{1}{2}t_2t_4.$$

Let A, B, C, D (fig. 1) be the poles of the four circles, O that of the fifth, a, b, c, d the points at which the four circles touch the fifth.



From t_1, t_2 draw arcs of great circles through A, B, and produce them till they meet at N; then Nt_1 and Nt_2 are quadrants.

Put arc $Oa=R$, $Aa=r_1$, $Bb=r_2$, $AB=d$. Then in the spherical triangle Oab we have

$$\sin\tfrac{1}{2}ab=\sin R\sin\tfrac{1}{2}aOb. \tag{1}$$

In the triangle AOB we have

$$\sin^2\tfrac{1}{2}aOb=\frac{\cos(r_1-r_2)-\cos d}{2\sin(R+r_1)\sin(R+r_2)}. \tag{2}$$

In the triangle ANB we have

$$\sin^2\tfrac{1}{2}t_1t_2=\sin^2\tfrac{1}{2}N=\frac{\cos(r_1-r_2)-\cos d}{2\cos r_1\cos r_2}. \tag{3}$$

From (1), (2), and (3) we readily find

$$\sin\tfrac{1}{2}ab=\sin\tfrac{1}{2}t_1t_2\sin R\sqrt{\left[\frac{\cos r_1\cos r_2}{\sin(R+r_1)\sin(R+r_2)}\right]}.$$

In like manner we can find the value of $\sin\frac{1}{2}bc$, $\sin\frac{1}{2}cd$, &c.. But since the chords joining the points a, b, c, d, form the sides and diagonals of an inscribed quadrilateral, and each chord is twice the sine of half the corresponding arc, we have

$$\sin\tfrac{1}{2}ab\sin\tfrac{1}{2}cd+\sin\tfrac{1}{2}ad\sin\tfrac{1}{2}bc=\sin\tfrac{1}{2}ac\sin\tfrac{1}{2}bd. \tag{4}$$

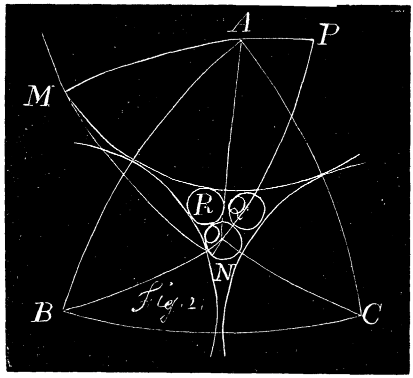
Substituting the values of $\sin\frac{1}{2}ab$, $\sin\frac{1}{2}bc$, &c., in (4), and omitting the common factor

$$\sin^2 R\sqrt{\left[\frac{\cos r_1\cos r_2\cos r_3\cos r_4}{\sin(R+r_1)\sin(R+r_2)\sin(R+r_3)\sin(R+r_4)}\right]},$$

we have

$$\sin\tfrac{1}{2}t_1t_2\sin\tfrac{1}{2}t_3t_4+\sin\tfrac{1}{2}t_2t_3\sin\tfrac{1}{2}t_4t_1=\sin\tfrac{1}{2}t_1t_3\sin\tfrac{1}{2}t_2t_4. \tag{5}$$

Let A, B, C (fig. 2) be the poles of the small circles, radii a, b, c; O, Q, R the poles of the small circles, radii x_1, y_1, z_1, MN the arc of a great circle which joins the circles A and O. From M and N draw arcs of great circles through A and O, and produce them till they meet at P.



Let t_1, t_2, t_3 be the lengths of the arcs of great circles which touch the circles A and O, B and Q, C and R respectively.

Put arc $AB = a + b$, $AC = a + c$, $BC = b + c$, $BO = b + x_1$, $CO = c + x_1$, $AO = s$, $MN = t_1$, $\angle CBO = \theta$, $\angle ABO = \phi$, $\angle ABC = \beta$.

Since each of the circles O, Q, R, is touched by four circles which touch each other consecutively we have, by (5),

$$\sin\tfrac{1}{2}t_1 \sin\tfrac{1}{2}t_2 = \surd[(\tan a \tan b)(\tan x_1 \tan y_1)] + \surd[(\tan a \tan y_1)(\tan b \tan x_1)]$$

$$= 2\surd(\tan a \tan b \tan x_1 \tan y_1), \qquad (6)$$

$$\sin\tfrac{1}{2}t_1 \sin\tfrac{1}{2}t_3 = 2\surd(\tan a \tan c \tan x_1 \tan z_1), \qquad (7)$$

$$\sin\tfrac{1}{2}t_2 \sin\tfrac{1}{2}t_3 = 2\surd(\tan b \tan c \tan y_1 \tan z_1). \qquad (8)$$

From (6), (7) and (8) we find

$$\sin\tfrac{1}{2}t_1 = \surd(2 \tan a \tan x_1). \qquad (9)$$

$$\therefore \quad \cos t_1 = 1 - 2\sin^2\tfrac{1}{2}t_1 = 1 - 4\tan a \tan x_1. \qquad (10)$$

In the spherical triangle AOP we have

$$\cos s = \sin a \sin x_1 + \cos a \cos x_1 \cos t_1 = \cos a \cos x_1 - 3\sin a \sin x_1. \quad (11)$$

In the triangle ABO we have

$$\cos\varphi = \frac{\cos s - \cos(a+b)\cos(b+x_1)}{\sin(a+b)\sin(b+x_1)} = 1 - \frac{4\operatorname{cosec}^2 b}{(\cot a \cot b)(\cot x_1 \cot b)};$$

$$\therefore \quad \sin\tfrac{1}{2}\varphi = \frac{\surd 2.\operatorname{cosec} b}{\surd[(\cot a + \cot b)(\cot x_1 + \cot b)]}. \qquad (12)$$

Similarly we find

$$\sin\tfrac{1}{2}\theta = \frac{\operatorname{cosec} b}{\surd[(\cot b + \cot c)(\cot x_1 + \cot b)]}, \qquad (13)$$

$$\sin\tfrac{1}{2}\beta = \frac{\operatorname{cosec} b}{\surd[(\cot a + \cot b)(\cot b + \cot c)]}, \qquad (14)$$

$$\cos\tfrac{1}{2}\beta = \sqrt{\left[\frac{\cot a \cot b + \cot a \cot c + \cot b \cot c - 1}{(\cot a + \cot b)(\cot b + \cot c)}\right]}. \qquad (15)$$

But $\sin\frac{1}{2}(\theta + \varphi) = \sin\frac{1}{2}\beta$, whence by developing, transposing, squaring, and reducing, we find

$$\sin^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\varphi - \sin^2\tfrac{1}{2}\beta + 2\cos\tfrac{1}{2}\beta \sin\tfrac{1}{2}\theta \sin\tfrac{1}{2}\varphi = 0. \qquad (16)$$

Substituting the values of $\sin\frac{1}{2}\theta$, &c., in (16), clearing and reducing, we get an equation of the first degree, whence we readily find

$$\cot x_1 = \cot a + 2\cot b + 2\cot c + 2\surd(2\cot a \cot b + 2\cot a \cot c + 2\cot b \cot c - 2).$$

Similarly we have

$\cot y_1 = 2\cot a + \cot b + 2\cot c + 2\sqrt{(2\cot a \cot b + 2\cot a \cot c + 2\cot b \cot c - 2)}$,

$\cot z_1 = 2\cot a + 2\cot b + \cot c + 2\sqrt{(2\cot a \cot b + 2\cot a \cot c + 2\cot b \cot c - 2)}$.

Put $p_n = 2\cot x_n + 2\cot y_n + 2\cot z_n + 2\sqrt{(2\cot x_n \cot y_n + 2\cot x_n \cot z_n + 2\cot y_n \cot z_n - 2)}$,

and $q_n = 2\cot x_n + 2\cot y_n + 2\cot z_n$. Then we have

$$\cot x_1 = p - \cot a, \quad \cot y_1 = p - \cot b, \quad \cot z_1 = p - \cot c. \qquad (17)$$

By substitution we find $p_1 = 11p - 2q$, $q_1 = 6p - q$, and similarly, $p_2 = 11p_1 - 2q_1$, $q_2 = 6p_1 - q_1$. From these relations we find

$$p_2 - 10p_1 + p = 0, \qquad (e_1)$$

and similarly,

$$p_3 - 10p_2 + p_1 = 0, \qquad (e_2)$$

$$p_4 - 10p_3 + p_2 = 0, \qquad (e_3)$$

$$\cdots\cdots\cdots$$

$$p_{n+1} - 10p_n + p_{n-1} = 0. \qquad (e_n)$$

From (17) we have $\cot x_1 + \cot a = \cot y_1 + \cot b = \cot z_1 + \cot c = p$,

and similarly $\cot x_2 + \cot x_1 = \cot y_2 + \cot y_1 = \cot z_2 + \cot z_1 = p_1$,

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cot x_n + \cot x_{n-1} = \cot y_n + \cot y_{n-1} = \cot z_n + \cot z_{n-1} = p_{n-1}.$$

Subtracting the fiirst of these equations from the second, the resulting equation from the third, and so on to the last, we find

$$\cot x_n \pm \cot a = \cot y_n \pm \cot b = \cot z_n \pm \cot c = p_{n-1} - p_{n-2} + p_{n-3} - \ldots \pm p, \qquad (18)$$

the double sign being taken plus, when n is odd, and minus when n is even.

Let $u_n = \cot x_n \pm \cot a = \cot y_n \pm \cot b = \cot z_n \pm \cot c$. Then we have

$$u_n = p_{n-1} - p_{n-2} + p_{n-3} - \ldots \pm p, \qquad (19)$$

$$u_{n+1} = p_n - p_{n-1} + p_{n-2} - \ldots \pm p_1 \pm p, \qquad (20)$$

$$u_{n+2} = p_{n+1} - p_u + p_{n-1} - \ldots \pm p_2 \pm p_1 \pm p. \qquad (21)$$

Adding (19) and (21), subtracting (20)$\times 10$ from the resulting equation, and substituting from (e_1), (e_2), (e_3) ... (e_n), we find

$$u_{n+2} - 10u_{n+1} + u_n = \pm 2q, \qquad (22)$$

and similarly we get

$$u_{n+3} - 10u_{n+2} + u_{n+1} = \mp 2q. \qquad (23)$$

Adding (22) and (23), $u_{n+3} - 9u_{n+2} + 9u_{n+1} - u_n = 0$, (24)

an equation in Finite Differences.

Integrating (24), $u_n = C_1(5+2\sqrt{6})^n + C_2(5-2\sqrt{6})^n + C_3(-1)^n$. (25)

When $n = 0$, $\quad u_0 = C_1 + C_2 + C_3 = 0$. (26)

" $n = 1$, $\quad u_1 = C_1(5+2\sqrt{6}) + C_2(5-2\sqrt{6}) - C_3 = p$. (27)

" $n = 2$, $\quad u_n = C_1(5+2\sqrt{6})^2 + C_2(5-2\sqrt{6})^2 + C_3 = 10p - 2q$. (28)

From (26), (27), and (28) we find $C_1 = \frac{1}{24}p\sqrt{6} - \frac{1}{24}q(\sqrt{6}-2)$,

$C_2 = -\frac{1}{24}p\sqrt{6} + \frac{1}{24}q(\sqrt{6}+2)$, and $C_3 = -\frac{1}{6}q$.

Substituting in (25) we find $\cot x_n \pm \cot a = \cot y_n \pm \cot b = \cot z_n \pm \cot c$

$= \frac{1}{6}[(5+2\sqrt{6})^n + (5-2\sqrt{6})^n \pm 2]\,(\cot a + \cot b + \cot c)$

$+ \frac{1}{6}[(5+2\sqrt{6})^n - (5-2\sqrt{6})^n\quad]\sqrt{(3\cot a \cot b + 3\cot a \cot c + 3\cot b \cot c - 3)}$.